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Amemiya norm equals Orlicz norm in general*

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ABSTRACT

We present a proof that in Orlicz spaces the Amemiya norm and the Orlicz norm coincide for any Orlicz function φ. This gives the answer for an open problem. We also give a description of the Amemiya type for the Mazur-Orlicz F-norm.

1. INTRODUCTION

In the theory of Orlicz spaces three norms appeared in the following historical way: First it was Orlicz who introduced already in thirties the so called *Orlicz norm* $\|\cdot\|_\varphi^0$.

Then Nakano (1950), Morse-Transue (1950) and Luxemburg (1955) considered another norm, which sometimes is called the Luxemburg-Nakano norm but usually in the literature is called the *Luxemburg norm*. This norm is just the Minkowski functional of a convex modular ball $\{x : I_\varphi(x) \leq 1\}$, i.e.,

$$\|x\|_\varphi = \inf\{\lambda > 0 : I_\varphi(x/\lambda) \leq 1\}.$$

Approximatively at the same time I. Amemiya (see [11], p. 218) considered the norm

$$\|x\|_\varphi^A = \inf_{k>0} \frac{1}{k}(1 + I_\varphi(kx)).$$

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In the Nakano book [11] there is a discussion about two kinds of norms, the so called first norm which is the Amemiya norm and the second norm $|||x|||_\varphi = \inf\left\{\frac{1}{|\alpha|} : I_\varphi(\alpha x) \le 1\right\}$, which is, as we can easily see, the Luxemburg norm. We have also the following relation between these norms

$$\|x\|_\varphi \le \|x\|_\varphi^A \le 2\,\|x\|_\varphi$$

for every $x \in L_\varphi(\mu)$. A similar relation, where instead of the Amemiya norm we put the Orlicz norm, is also well known. Namely, for any $x \in L_\varphi(\mu)$,

$$\|x\|_\varphi \le \|x\|_\varphi^0 \le 2\,\|x\|_\varphi.$$

Moreover, $\|x\|_\varphi \le 1$ if and only if $I_\varphi(x) \le 1$. For the theory of Orlicz spaces we refer to [3], [4], [5], [6], [9], [11], [13] and [14].

The last two relations between the norms suggested that maybe the Orlicz norm and the Amemiya norm are equal. Really, the equality

$$(1) \qquad \|x\|_\varphi^0 = \|x\|_\varphi^A$$

for any $x \in L_\varphi(\mu)$, when φ is an N-function (i.e., φ is a finite-valued vanishing only at zero Orlicz function which satisfies the conditions $\varphi(u)/u \to 0$ as $u \to 0^+$ and $(\varphi(u)/u \to \infty$ as $u \to \infty)$ was proved by Krasnoselskii-Rutickii in [3], Th. 10.5 (see also [7], Th. 8.6 or [13], Th. 13). Nakano [11] and Luxemburg-Zaanen [6] have shown the equality (1) for every Orlicz function φ which is finite-valued and satisfies the condition $\varphi(u)/u \to \infty$ as $u \to \infty$. As far as we know the problem if the norms $\|\cdot\|_\varphi^0$ and $\|\ \|_\varphi^A$ coincide in any Orlicz space $L_\varphi(\mu)$ (or equivalently for any Orlicz function φ) seems to be an open question.

We should mention here that the equality

$$I_{\varphi *}(y) = \sup\left\{\left|\int_\Omega x(t)y(t)d\mu\right| - I_\varphi(x) : x \in L_\varphi\right\},$$

was used by Nakano [10] in order to prove equality of Orlicz and Amemiya norms. The desired equality follows from the Nakano result saying that in any abstract modular space the Amemiya norm coincides with the Orlicz norm if we can prove that on order continuous part of the dual to an Orlicz space L_φ the conjugate modular $(I_\varphi)^*$ coincides with the modular I_{φ^*}. However, to prove this equality we need to know that for any $u > 0$ there is $v > 0$ such that the Young equality $\varphi(u) + \varphi^*(v) = uv$ holds. But for some Orlicz functions this is not true. Namely, if we take as the Orlicz funtion

$$\varphi(u) = \begin{cases} 1 - \sqrt{1-u^2} & \text{for } 0 \le u \le 1, \\ \infty & \text{for } u > 1. \end{cases}$$

Then the Young conjugate φ^* is

$$\varphi^*(v) = \sqrt{1+v^2} - 1,$$

and for $u = 1$ there is no v satisfying the Young equality.

Similarly, the proof in [6] works for any convex Orlicz function φ such that at any point at which $\varphi(u) < \infty$ its left-derivative is also finite. The above example however shows that it can be $\varphi'_{-}(b(\varphi)) = \infty$.

For those reasons the problem of equality of the Amemiya and Orlicz norms needs an independent proof.

We also note that although L_1, L_∞, $L_1 \cap L_\infty$ and $L_1 + L_\infty$ are generated by Orlicz functions which are neither finite-valued nor vanishing only at zero, it is known that both of these norms also coincide in such spaces (see [2]).

In this paper we will solve the above mentioned problem by showing that the Amemiya norm and the Orlicz norm are equal for any Orlicz function φ. The proof of this result is given in Section 2. In Section 1 we collect some necessary definitions and some known results from the theory of Orlicz spaces. Finally, in Section 3 there are some results and remarks in connection to the Amemiya type description of the norm. Already Orlicz [12] observed that the Luxemburg norm can be described in the Amemiya form. We will see that also the Mazur-Orlicz F-norm can be described in the Amemiya form.

2. PRELIMINARIES

Let (Ω, Σ, μ) be a complete σ-finite measure space and $L^0(\mu)$ the space of all (equivalence classes of) Σ-measurable real functions on Ω. Consider an *Orlicz function* $\varphi : [0,\infty) \to [0,\infty]$, i.e., a *convex function vanishing at zero* (not identically 0 or ∞ on $(0,\infty)$) and define a functional $I_\varphi : L^0(\mu) \to [0,\infty]$ by the formula $I_\varphi(x) = \int_\Omega \varphi(|x(t)|)\ \mathrm{d}\mu$. The *Orlicz space* $L_\varphi(\mu)$ is defined by

$$L_\varphi(\mu) = \{x \in L^0(\mu) : I_\varphi(\lambda x) < \infty \text{ for some } \lambda > 0\}.$$

This space is a Banach space with the following three norms: the *Luxemburg norm*

$$\|x\|_\varphi = \inf\{\lambda > 0 : I_\varphi(x/\lambda) \le 1\},$$

the *Amemiya norm*

$$\|x\|_\varphi^A = \inf_{k>0} \frac{1}{k}(1 + I_\varphi(kx))$$

and the Orlicz norm

$$\|x\|_\varphi^0 = \sup\left\{\left|\int_\Omega x(t)y(t)d\mu\right| : y \in L_{\varphi*}, I_{\varphi*}(y) \le 1\right\},$$

where the function $\varphi^* : [0,\infty) \to [0,\infty]$, defined by the formula

$$\varphi^*(u) = \sup\{uv - \varphi(v) : v \ge 0\},$$

is called the *complementary function* to φ in the sense of Young (see [3], [7], [9], [13]). We need to define for an Orlicz function φ the following four numbers:

$$a(\varphi) = \sup\{u \geq 0 : \varphi(u) = 0\}, \quad b(\varphi) = \sup\{u > 0 : \varphi(u) < \infty\},$$
$$c(\varphi) = \lim_{u \to 0^+} (\varphi(u)/u) = \lim_{u \to 0^+} \varphi'_-(u) = \lim_{u \to 0^+} \varphi'_+(u),$$
$$d(\varphi) = \lim_{u \to \infty} (\varphi(u)/u) = \lim_{u \to \infty} \varphi'_-(u) = \lim_{u \to \infty} \varphi'_+(u),$$

where $\varphi'_-(u)$ and $\varphi'_+(u)$ denote the left and the right derivatives of φ at u, respectively. From the definition of the Orlicz function we have $a(\varphi) \leq b(\varphi)$, $a(\varphi) < \infty$, $b(\varphi) > 0$ and $0 \leq c(\varphi) \leq \varphi(1) \leq d(\varphi)$. Moreover, $a(\varphi^*) = c(\varphi)$ and $b(\varphi^*) = d(\varphi)$.

The Orlicz space $L_\varphi(\mu)$ with each of the above three norms $\|\cdot\|$ is a symmetric space with the Fatou property (see [3], [7] and [14]). More precisely, they are *Banach function lattices*, i.e.,

(i) if $x \in L^0(\mu), y \in L_\varphi(\mu)$ and $|x(t)| \leq |y(t)|$ μ-a.e., then $x \in L_\varphi(\mu)$ and $\|x\| \leq \|y\|$.

(ii) there is a function $x \in L_\varphi(\mu)$ which is positive μ-a.e. They also have the Fatou property, i. e.,

(iii) if $0 \leq x_n \uparrow x$ with $x_n \in L_\varphi(\mu)$, $x \in L^0(\mu)$ and $\sup_n \|x_n\| < \infty$, then $x \in L_\varphi(\mu)$ and

$$\|x\| = \lim_{n \to \infty} \|x_n\|,$$

which are also *symmetric*, i.e.,

(iv) if $x \in L^0(\mu)$, $y \in L_\varphi(\mu)$ and $\mu\{t \in \Omega : |x(t)| > \lambda\} = \mu\{t \in \Omega : |y(t)| > \lambda\}$ for all $\lambda > 0$, then $x \in L_\varphi(\mu)$ and $\|x\| = \|y\|$.

The Fatou property for the Amemiya norm is not obvious and therefore we give a proof here. Assume that $0 \leq x_n \uparrow x$ with $x_n \in L_\varphi(\mu)$, $x \in L^0(\mu)$ and $\sup_n \|x_n\|_\varphi^A < \infty$. Then

$$\alpha_n := \|x_n\|_\varphi^A = \inf_{k>0} \frac{1 + I_\varphi(kx_n)}{k} \leq \inf_{k>0} \frac{1}{k}(1 + I_\varphi(kx)) = \|x\|_\varphi^A,$$

and so $\alpha = \sup \alpha_n \leq \|x\|_\varphi^A$. We want to prove that $\|x\|_\varphi^A \leq \alpha$.

Assume that the last inequality is not true, i.e., $\|x\|_\varphi^A > \alpha$. Then $\|x\|_\varphi^A > \alpha + \delta$ for some $\delta > 0$, and this gives that $\frac{1}{k}[1 + I_\varphi(kx)] \geq \alpha + \delta$ for any $k > 0$. Since, by the Fatou lemma for integrals, we have $I_\varphi(kx) \leq \liminf_{n \to \infty} I_\varphi(kx_n)$ it follows that

$$\frac{1}{k}[1 + I_\varphi(kx_{n_m})] \geq \frac{1}{k}[1 + I_\varphi(kx)] \geq \alpha + \delta,$$

for some subsequence (x_{n_m}) of (x_n). This gives $\|x_{n_m}\|_\varphi^A \geq \alpha + \delta > \alpha$ for all natural numbers m, which is a contradiction. The proof is complete.

The Orlicz function φ and its complementary function φ^* satisfy the Young inequality

(2) $\qquad uv \leq \varphi(u) + \varphi^*(v) \; \forall u, v \geq 0.$

The case of equality in (2) is important and leads to the definition of the *subdifferential* $\partial\varphi(u)$ of φ at $u \geq 0$ as follows:

$$\partial\varphi(u) = \{v \geq 0 : \varphi(u) + \varphi^*(v) = uv\}.$$

Observe that:

(i) if $u \in (0, b(\varphi))$ and $\varphi'_-(u) < \infty$, then $\partial\varphi(u) = [\varphi'_-(u), \varphi'_+(u)]$;

(ii) if $\varphi'_-(b(\varphi)) < \infty$, then $\partial\varphi(b(\varphi)) = [\varphi'_-(b(\varphi)), \infty)$;

(iii) if either $u = b(\varphi)$ and $\varphi'_-(u) = \infty$ or $u > b(\varphi)$, then $\partial\varphi(u) = \emptyset$.

Since φ is defined only on $[0, \infty)$, $\varphi'_-(0)$ is not defined, we set $\partial\varphi(0) = [-\varphi'_+(0), \varphi'_+(0)]$.

Note that if we extend φ to the whole R, by letting $\varphi(-u) = \varphi(u)$ for all $u \geq 0$, then we obviously have that $\varphi'_-(-u) = -\varphi'_+(u)$ and $\varphi'_+(-u) = \varphi'_-(u)$.

3. THE MAIN RESULT

We formulate and prove now our main result.

Theorem 1. *Let φ be an arbitrary Orlicz function. Then*

$$\|x\|^0_\varphi = \|x\|^A_\varphi$$

for any $x \in L_\varphi(\mu)$.

Proof. Using the Young inequality (2) we obtain for any $k > 0$ and $y \in L_{\varphi^*}$ with $I_{\varphi^*}(y) \leq 1$,

$$\left|\int_\Omega x(t)y(t)d\mu\right| = \frac{1}{k}\left|\int_\Omega kx(t)y(t)d\mu\right|$$

$$\leq \frac{1}{k}\{I_\varphi(kx) + I_{\varphi^*}(y)\} \leq \frac{1}{k}\{I_\varphi(kx) + 1\}.$$

This gives $\| x \|^0_\varphi \leq \frac{1}{k}\{I_\varphi(kx) + 1\}$ $\forall k > 0$, and, consequently,

$$\|x\|^0_\varphi \leq \|x\|^A_\varphi$$

for any $x \in L_\varphi(\mu)$. It only remains to prove the reversed inequality

$$(3) \qquad \|x\|^A_\varphi \leq \|x\|^0_\varphi$$

for any $x \in L_\varphi(\mu)$.

Let us first observe that to prove (3) in general it is really sufficient to show (3) when

$(*)$ x is a nonnegative simple function from $L_\varphi(\mu)$ and φ is a smooth Orlicz function on $[0, b(\varphi))$.

Really, according to the fact that both of the norms $\| \, \|^A_\varphi$ and $\| \cdot \|^0_\varphi$ have the Fatou property, it is sufficient to show (3) for simple functions from $L_\varphi(\mu)$. Moreover, the norms are ideal ones, i.e., we have the equalities

$$\|x\|^0_\varphi = \| |x| \|^0_\varphi \text{ and } \|x\|^A_\varphi = \| |x| \|^A_\varphi,$$

so it is sufficient to prove our estimate (3) only for nonnegative simple functions from $L_\varphi(\mu)$.

Also without loss of generality we can assume that φ is smooth on the interval $[0, b(\varphi))$. Otherwise, for any $\varepsilon > 0$ we can find an Orlicz function ψ which is smooth on $[0, b(\psi))$ and satisfying

(4) $$\psi(u) \le \psi(u) \le \psi((1+\varepsilon)u) \quad \forall u > 0.$$

Take, e.g., $\varphi_\varepsilon(u) = \frac{1}{\ln(1+\varepsilon)} \int_u^{(1+\varepsilon)u} \frac{\varphi(s)}{s}\, ds$. Then we have $\varphi(u) \le \varphi_\varepsilon(u) \le \varphi((1+\varepsilon)u)$ for all $u \ge 0$ and so $L_{\varphi_\varepsilon}(\mu) = L_\varphi(\mu)$ with

(5) $$\|x\|_\varphi^0 \le \|x\|_{\varphi_\varepsilon}^0 \le (1+\varepsilon)\|x\|_\varphi^0.$$

If we can prove the equality of the Amemiya and the Orlicz norm for smooth functions φ on $[0, b(\varphi))$, then we have $\|x\|_\psi^0 = \|x\|_\psi^A$ for all $x \in L_\psi(\mu) = L_\varphi(\mu)$. Moreover, by (4),

$$\begin{aligned}&\inf_{k>0} \frac{1}{k}(1 + I_\varphi(kx)) \le \inf_{k>0} \frac{1}{k}(1 + I_\psi(kx)) \le \inf_{k>0} \frac{1}{k}(1 + I_\varphi((1+\varepsilon)kx)) \\ &= (1+\varepsilon) \inf_{k>0} \frac{1}{k(1+\varepsilon)}(1 + I_\varphi((1+\varepsilon)kx)),\end{aligned}$$

and, thus, we find that

(6) $$\|x\|_\varphi^A \le \|x\|_\psi^A \le (1+\varepsilon)\|x\|_\varphi^A.$$

In virtue of (5) and (6), we obtain

$$\|x\|_\varphi^0 \le \|x\|_\varphi^A \le \|x\|_\psi^A = \|x\|_\psi^0 \le (1+\varepsilon)\|x\|_\varphi^0,$$

so that

$$\|x\|_\varphi^0 \le \|x\|_\varphi^A \le (1+\varepsilon)\|x\|_\varphi^0$$

and since $\varepsilon > 0$ is arbitrary we conclude that $\|x\|_\varphi^0 = \|x\|_\varphi^A$ for all $x \in L_\varphi(\mu)$. Observe also that

$$I_{\varphi^*}(\varphi'(kx)) \to 0 \text{ as } k \to 0$$

whenever φ is smooth, i.e., φ' is continuous (φ' denotes the right or left derivative of φ) and x is a nonnegative simple function from $L_\varphi(u)$. This follows from the Lebesgue dominated convergence theorem and the facts that

$$\varphi^*(\varphi'(u)) = u\varphi'(u) - \varphi(u) \to 0 \text{ as } u \to 0^+$$

and $\varphi^*(\varphi'(u))$ is nondecreasing on $[0, \infty)$.

Finally, the task is to prove (3) under restrictions $(*)$. We will consider the following three cases separately.

I. The function φ is finite-valued and $d(\varphi) = \infty$.

II. $d(\varphi) < \infty$.

III. $b(\varphi) < \infty$.

Case I: We have $\varphi(u)/u \to \infty$ as $u \to \infty$ and, by smoothness of φ on $[0, \infty)$, $\varphi'(u) \to \infty$ as $u \to \infty$. Also, for a nonnegative simple function x from $L_\varphi(\mu)$,

$$I_{\varphi^*}(\varphi'(kx)) < \infty \text{ for any } k > 0 \text{ and } I_{\varphi^*}(\varphi'(kx)) \to \infty \text{ as } k \to \infty.$$

Since the function $f(k) = I_{\varphi^*}(\varphi'(kx))$ is continuous, it has the Darboux property, so $f(k) = I_{\varphi^*}(\varphi'(kx)) = 1$ for some $k > 0$. This yields

$$\begin{aligned} \|x\|_\varphi^A &\le \frac{1}{k}\{1 + I_\varphi(kx)\} = \frac{1}{k}\{I_{\varphi^*}(\varphi'(kx)) + I_\varphi(kx)\} \\ &= \frac{1}{k}\int_\Omega kx(t)\varphi'(kx(t))d\mu = \int_\Omega x(t)\varphi'(kx(t))d\mu \le \|x\|_\varphi^0. \end{aligned}$$

Case II. $d(\varphi) < \infty$. Then $b(\varphi) = \infty$ and $b(\varphi^*) = d(\varphi)$.

It is also obvious that for smooth φ, we have $\varphi'(u) \uparrow d(\varphi)$ as u $\uparrow \infty$. We will consider two subcases for a nonnegative simple function $x \in L_\varphi(\mu)$.

IIa. $\mu(\text{suppx}) > 1/\varphi^*(d(\varphi))$.

Then there exists $\delta > 0$ such that $\mu(\text{suppx}) > 1/\varphi^*(d(\varphi) - \delta)$. Take any $\varepsilon \in (0, \delta]$ and let $u_\varepsilon > 0$ be so large that $\varphi'(u) \ge d(\varphi) - \varepsilon$ for all $u \ge u_\varepsilon$. Define

$$A_n = \{t \in \Omega : x(t) > u_\varepsilon/n\}.$$

There exists $m \in N$ such that $\mu(A_m) > 1/\varphi^*(d(\varphi) - \varepsilon)$. We have for any $t \in A_m$ that $mx(t) > u_\varepsilon$, and thus, $\varphi'(mx(t)) > \varphi'(u_\varepsilon) \ge d(\varphi) - \varepsilon$. Consequently, $\varphi^*[\varphi'(mx(t))] \ge \varphi^*[d(\varphi) - \varepsilon]$ and so

$$\infty > I_{\varphi^*}(\varphi'(mx)) \ge \varphi^*[d(\varphi) - \varepsilon]\mu(A_m) > \varphi^*[d(\varphi) - \varepsilon]/\varphi^*[d(\varphi) - \varepsilon] = 1.$$

Now, in the same way as in the case I, we note that there exists $k > 0$ such that $I_{\varphi^*}(\varphi'(kx)) = 1$ and, consequently, $\|x\|_\varphi^A \le \|x\|_\varphi^0$.

IIb. $\mu(\text{suppx}) \le 1/\varphi^*(d(\varphi))$.

Assume that $I_{\varphi^*}(y) \le 1$. Then $|y(t)| \le d(\varphi)$ μ-a.e. because $\varphi^*(u) = \infty$ for all $u > d(\varphi)$. This yields

$$\left|\int_\Omega x(t)y(t)d\mu\right| \le \int_\Omega |x(t)y(t)|d\mu \le d(\varphi)\int_\Omega |x(t)|d\mu = d(\varphi)\|x\|_1.$$

Setting $z(t) = d(\varphi)\chi_{\text{suppx}}(t)$, we get

$$\int_\Omega x(t)z(t)d\mu = d(\varphi)\|x\|_1 \text{ and } I_{\varphi^*}(z) = \varphi^*(d(\varphi))\mu(\text{suppx}) \le 1.$$

Thus, $\|x\|_\varphi^0 = d(\varphi)\|x\|_1$.

On the other hand,

$$\begin{aligned} \|x\|_\varphi^A &= \inf_{k>0}\frac{1}{k}(1 + I_\varphi(kx)) \le \lim_{k\to\infty}\frac{1}{k}\{1 + I_\varphi(kx)\} \\ &= \lim_{k\to\infty}\frac{1}{k}I_\varphi(kx) = \lim_{k\to\infty}\int_{\text{suppx}} \frac{|x(t)|}{k|x(t)|}\varphi(k|x(t)|)d\mu \\ &= \int_{\text{suppx}} |x(t)|\lim_{k\to\infty}\frac{\varphi(|kx(t)|)}{k|x(t)|}d\mu = d(\varphi)\|x\|_1 = \|x\|_\varphi^0. \end{aligned}$$

Therefore, $\|x\|_\varphi^A = \|x\|_\varphi^0$ and the proof in the case II is complete.

Case III: We consider again two subcases.

IIIa. $b(\varphi) < \infty$ and $\varphi'_{-}(b(\varphi)) = \infty$.

By the homogenity of the norms $\| \|_\varphi^A$ and $\| \cdot \|_\varphi^0$ it is enough to prove the equality of these norms only for $x \in L_\varphi(\mu)$ with $\|x\|_\varphi^A = 1$.

Let x be a nonnegative simple function $x = \sum_{i=1}^n c_i \chi_{A_i}$ with $\|x\|_\varphi^A = 1$, where $c_i \neq c_j$ for $i \neq j$ and A_i are pairwise disjoint measurable sets and $0 < \mu(A_i) < \infty$ for these $i \in \{1, \dots, n\}$ for which $\varphi(c_i) > 0$ (note that if $\varphi(c_i) = 0$ it can be $\mu(A_i) = \infty$).

Let m be such that $c_m = \max\{c_i : i = 1, \dots, n\}$. The function φ is finite-valued on $[0, b(\varphi))$ and $\varphi'(u) \uparrow \infty$ as $u \uparrow b(\varphi)$. Therefore there exists $0 < k < b(\varphi)/c_m$ such that $\varphi^*((\varphi'(kc_m))\mu(A_m) > 1$.

Moreover, $0 \leq \varphi^*(\varphi'(kx(t))) \leq \varphi^*(\varphi'(kc_m)) < \infty$ for any $t \in \Omega$ and this gives

$$1 < I_{\varphi^*}(\varphi'(kx)) < \infty.$$

Since $I_{\varphi^*}(\varphi'(\lambda x)) \to 0$ as $\lambda \to 0$ and the function $f(\lambda) = I_{\varphi^*}(\varphi'(\lambda x))$ is continuous on the interval $[0, b(\varphi)/c_m]$, it follows that there exists $k_0 \in (0, k)$ such that $I_{\varphi^*}(\varphi'(k_0 x)) = 1$, which yields

$$\|x\|_\varphi^A \leq \|x\|_\varphi^0.$$

IIIb. $b(\varphi) < \infty$ and $\varphi'_{-}(b(\varphi)) < \infty$.

Let x and c_m be defined as in the case IIIa. Note that $0 < \mu(A_m) < \infty$ when $\varphi(c_m) > 0$. Let $k = b(\varphi)/c_m$. If $I_{\varphi^*}(\varphi'(kx)) \geq 1$, where $\varphi'(b(\varphi))$ denotes $\varphi'_{-}(b(\varphi))$, then we can find as above a number $\lambda \in (0, k]$ such that $I_{\varphi^*}(\varphi'(\lambda x)) = 1$, which gives $\|x\|_\varphi^A \leq \|x\|_\varphi^0$.
Assume now that $I_{\varphi^*}(\varphi'(kx)) < 1$. By the definition of k, we have $kx(t) = b(\varphi)$ for all $t \in A_m$. Let A be a measurable subset of A_m of a positive and finite measure. Since φ^* is finite-valued, we can find a number $\beta > \varphi'_{-}(b(\varphi))$ such that

$$I_{\varphi^*}(\varphi'(kx)\chi_{\Omega \setminus A}) + \varphi^*(\beta)\mu(A) = 1.$$

Defining

$$y(t) = \varphi'(kx(t))\chi_{\Omega \setminus A}(t) + \beta \chi_A(t),$$

we have $y(t) \in \partial\varphi(kx(t))$ for μ-a.e. $t \in \Omega$ and $I_{\varphi^*}(y) = 1$.

Therefore $\|x\|_\varphi^A = \|x\|_\varphi^0$ if $\|x\|_\varphi^A = 1$ and so the equality holds for all $x \in L_\varphi(\mu)$.

We present now examples of Orlicz functions satisfying the assumptions from all of the cases I, II, IIIa and IIIb in the proof of Theorem 1. Assume below, in all the examples, that $1 \leq p < \infty$.

Example 1. Examples of finite-valued Orlicz functions with $d(\varphi) = \infty$ are (cf. case I):

$$\varphi(u) = u^p; \varphi_1(u) = u^p \ln(1 + u); \varphi_2(u) = \max\{0, u^p - 1\}, p > 1.$$

Note that $L_\varphi(\mu) = L_p(\mu), L_{\varphi_2}(\mu) = L_p(\mu) + L_\infty(\mu)$.

Example 2. Examples of Orlicz functions with $d(\varphi) < \infty$ are (cf. case II):

$$\varphi(u) = u; \varphi_1(u) = \begin{cases} u^p & \text{for } 0 \le u \le 1, \\ pu + 1 - p & \text{for } u \ge 1, \end{cases} \qquad \varphi_2(u) = \sqrt{1+u^2} - 1;$$
$$\varphi_3(u) = \max\{0, u-1\}.$$

Note that $L_\varphi(\mu) = L_1(\mu), L_{\varphi_1}(\mu) = L_p(\mu) + L_1(\mu), L_{\varphi_3}(\mu) = L_1(\mu) + L_\infty(\mu)$ and $\|x\|_\varphi = \|x\|_\varphi^0 = \|x\|_1, \|x\|_{\varphi_3}^0 = \|x\|_{\varphi_3}^A = \|x\|_{L_1 + L_\infty} = \inf\{\|u\|_1 + \|v\|_\infty : x = u + v,\ u \in L_1, v \in L_\infty\}$. Also $\|\cdot\|_{\varphi_3} \neq \|\cdot\|_{\varphi_3}^A$. For example, if $x_a(t) = 2a\chi_{[0,1/2]}(t) + a\chi_{[1,3]}(t), a > 0$, then $\|x_a\|_{\varphi_3} = \frac{6}{7}a < \frac{3}{2}a = \|x_a\|_{\varphi_3}^A$.

Example 3. The function

$$\varphi(u) = \begin{cases} 1 - \sqrt{1-u^2} & \text{for } 0 \le u \le 1, \\ \infty & \text{for } u > 1. \end{cases}$$

is an Orlicz function with $b(\varphi) < \infty$ and $\varphi'_-(b(\varphi)) = \infty$ (cf. case IIIa).

Example 4. Examples of Orlicz functions with $b(\varphi) < \infty$ and $\varphi'_-(b(\varphi)) < \infty$ are (cf. case IIIb):

$$\varphi(u) = \begin{cases} u^p & \text{for } 0 \le u \le 1, \\ \infty & \text{for } u > 1, \end{cases} \qquad \varphi_1(u) = \begin{cases} 0 & \text{for } 0 \le u \le 1, \\ \infty & \text{for } > 1. \end{cases}$$

Note that $b(\varphi) = b(\varphi_1) = 1, \varphi'_-(1) = p, (\varphi_1)'_-(1) = 0$. Moreover, $L_\varphi(\mu) = L_p(\mu) \cap L_\infty(\mu), L_{\varphi_1}(\mu) = L_\infty(\mu)$ and $\|x\|_\varphi = \max\{\|x\|_p, \|x\|_\infty\}, \|x\|_{\varphi_1} = \|x\|_{\varphi_1}^0 = \|x\|_\infty$.

We also note that

$$\|x\|_\varphi^A = \begin{cases} \beta(x)^{p-1}\|x\|_p + \|x\|_\infty & \text{if } \beta(x) \le (q/p)^{1/p}, \\ p^{1/p}q^{1/q}\|x\|_p & \text{if } \beta(x) \ge (q/p)^{1/p}, \end{cases} \tag{7}$$

where $\beta(x) = \|x\|_p/\|x\|_\infty$ for $x \neq 0$ and $1/p + 1/q = 1$. In particular, for $p = 1, \|x\|_\varphi^A = \|x\|_1 + \|x\|_\infty$.

Remark 1. Assuming that an Orlicz function φ satisfies $\varphi(u)/u \to \infty$ as $u \to \infty$ and that the measure μ is either non-atomic or counting, we can get that the infimum in the Amemiya formula is attained for a certain $k^* = k^*(x) > 0$, that is, $\|x\|_\varphi^A = \frac{1}{k^*}[1 + I_\varphi(k^* x)]$ (see [1], Lemma 1).

4. ADDITIONAL RESULTS AND REMARKS

Already Orlicz showed in 1961 that the Luxemburg norm can be described in the Amemiya form. This description shows well the difference between the Luxemburg norm and the Orlicz norm in $L_\varphi(\mu)$.

Theorem A (Orlicz [12]). *For any Orlicz function φ and any $x \in L_\varphi(\mu)$, we have*

$$\|x\|_\varphi = \inf_{k>0}\left\{\max\left(\frac{1}{k}, \frac{1}{k} I_\varphi(kx)\right)\right\}.$$

Remark 2. Orlicz showed in [12] even more, that is, if in the definition of an Orlicz function instead of convexity we have *s-convexity* ($0 < s \leq 1$; $s = 1$ is just a convexity), i.e., $\varphi(\alpha u + \beta v) \leq \alpha^s \varphi(u) + \beta^s \varphi(v)$ for all $u, v \geq 0, \alpha, \beta \geq 0$, $\alpha^s + \beta^s = 1$, then we can introduce an s-homogeneous norm by $\|x\|_{s\varphi} = \inf\{\lambda > 0 : I_\varphi(x/\lambda^{1/s}) \leq 1\}$ on $L_\varphi(\mu)$.

Orlicz proved also the following formula of the Amemiya type for this s-norm:

$$\|x\|_{s\varphi} = \inf_{k>0}\left\{\max\left(\frac{1}{k^s}, \frac{1}{k^s} I_\varphi(kx)\right)\right\}.$$

Remark 3. Theorem 1 and Orlicz's Theorem A suggest to consider the following family of the norms depending on p, $1 \leq p \leq \infty$,

$$\|x\|_{\varphi,p} = \inf_{k>0} \frac{[1 + I_\varphi(kx)^p]^{1/p}}{k}.$$

For $p = 1$ this is the Amemiya norm which is, by Theorem 1, equal to the Orlicz norm and for $p = \infty$ this is the Luxemburg norm. The norms $\|\cdot\|_{\varphi,p}$ decreases in p and

$$\|x\|_{\varphi,\infty} \leq \|x\|_{\varphi,p} \leq \|x\|_{\varphi,1} \leq 2^{1-1/p}\|x\|_{\varphi,p}.$$

There is a natural Köthe duality between these norms, that is, the Köthe dual of $\|\cdot\|_{\varphi,p}$ is $\|\cdot\|_{\varphi^*,q}$, where $1/p + 1/q = 1$.

Theorem A and Remark 2 motivates us to do similar description in Orlicz spaces which are F-spaces. Let $\Phi : [0,\infty) \to [0,\infty)$ be a *φ-function*, that is, a non-decreasing continuous vanishing at zero and unbounded function. Then the Orlicz space $L_\Phi(\mu)$ is defined again as the set of all $x \in L^0(\mu)$ such that $I_\Phi(\lambda x) = \int_\Omega \Phi(\lambda|x(t)|)d\mu < \infty$ for some $\lambda > 0$. The space $L_\Phi(\mu)$ is an F-space with the *Mazur-Orlicz F-norm*

$$|x|_\Phi = \inf\{\lambda > 0 : I_\Phi(x/\lambda) \leq \lambda\}$$

and with the *Koshi-Shimogaki F-norm*

$$|x|_\Phi^0 = \inf_{k>0}\left(\frac{1}{k} + I_\Phi(kx)\right).$$

Moreover, $|x|_\Phi \leq |x|_\Phi^0 \leq 2|x|_\Phi$ (see [7], Th. 1.1).

Theorem 2. *For any φ-function Φ and any $x \in L_\Phi(\mu)$, we have*

$$|x|_\Phi = \inf_{k>0} \max\left\{\frac{1}{k}, I_\Phi(kx)\right\}.$$

Proof. If $I_\Phi(kx) \leq 1/k$, then $|x|_\Phi \leq 1/k$ and if $I_\Phi(kx) > 1/k$, then $I_\Phi(\frac{x}{I_\Phi(kx)}) \leq I_\Phi(kx)$ and so $|x|_\Phi \leq I_\Phi(kx)$. Thus

$$|x|_{\Phi} \leq \max\{1/k, I_{\Phi}(kx)\}$$

for any $k > 0$ and we conclude that

$$|x|_{\Phi} \leq \inf_{k>0} \max\left\{\frac{1}{k}, I_{\Phi}(kx)\right\}.$$

On the other hand, if $0 < k_0 < 1/|x|_{\Phi}$, then $I_{\Phi}(k_0 x) \leq 1/k_0$, i.e.,

$$\inf_{k>0} \max\left\{\frac{1}{k}, I_{\Phi}(kx)\right\} \leq \max\left(\frac{1}{k_0}, I_{\Phi}(k_0 x)\right) \leq \frac{1}{k_0}.$$

Since $1/k_0$ can be arbitrarily close to $|x|_{\Phi}$, we get

$$\inf_{k>0} \max\left\{\frac{1}{k}, I_{\Phi}(kx)\right\} \leq |x|_{\Phi},$$

and the proof is complete.

Remark 4. Theorem A (or Orlicz result in Remark 1) and Theorem 2, that is the Amemiya type formula for the Luxemburg norm and also the corresponding formula for the Mazur-Orlicz F-norm, remains valid in any modular space $X_{\rho} = \{x \in X : \rho(\lambda x) \to 0 \text{ as } \lambda \to 0^+\}$, where X is an abstract linear space and ρ is either a convex (or s-convex) modular on X or only the modular on X, respectively (cf. [7], [9] for the definition of modular, convex or s-convex modular).

Appendix 1: Proof of (7)

If $x \in L_{\varphi}(\mu) = L_p(\mu) \cap L_{\infty}(\mu)$, then

$$\begin{aligned}\|x\|_{\varphi}^{A} &= \inf\left\{\frac{1}{k}[1 + I_{\varphi}(kx)] : k > 0, I_{\varphi}(kx) < \infty\right\} \\ &= \inf\left\{\frac{1}{k}[1 + k^p\|x\|_p^p] : 0 < k \leq 1/\|x\|_{\infty}\right\} = \inf\{f_x(k) : 0 < k \leq 1/\|x\|_{\infty}\}.\end{aligned}$$

The infimum of the function $f_x(k)$ on the interval $[0, 1/\|x\|_{\infty}]$ is attained at:

(a) $k_x = \left(\frac{q}{p}\right)^{1/p} \frac{1}{\|x\|_p}$ when $k_x \leq 1/\|x\|_{\infty}$ or equivalently if $\beta(x) \geq \left(\frac{q}{p}\right)^{1/p}$ and it is equal to

$$f_x(k_x) = \left\{\left(\frac{p}{q}\right)^{1/p} + \left(\frac{q}{p}\right)^{1/q}\right\}\|x\|_p = p^{1/p}q^{1/q}\|x\|_p,$$

(b) at $k_x = 1/\|x\|_{\infty}$ when $k_x \geq 1/\|x\|_{\infty}$ and it is equal to

$$f_x(k_x) = \|x\|_{\infty} + \|x\|_p^p\|x\|_{\infty}^{1-p} = \beta(x)^{p-1}\|x\|_p + \|x\|_{\infty}.$$

Thus, the equality (7) is proved.

Appendix 2: Proof of Theorem A

For the sake of completeness we will present two proofs of Orlicz's Theorem A.

Orlicz's proof. If $I_\varphi(kx) \le 1$, then $\|x\|_\varphi \le 1/k$. If $I_\varphi(kx) > 1$, then

$$I_\varphi\left(\frac{x}{k^{-1}I_\varphi(kx)}\right) = I_\varphi\left(\frac{kx}{I_\varphi(kx)}\right) \le I_\varphi(kx)/I_\varphi(kx) = 1,$$

whence $\|x\|_\varphi \le k^{-1}I_\varphi(kx)$. Therefore

$$\|x\|_\varphi \le \max(1/k, I_\varphi(kx)/k).$$

On the other hand, if $0 < k_0 < 1/\|x\|_\varphi$, then $I_\Phi(k_0x) \le 1$, i.e.,

$$\inf_{k>0}\left\{\max\left(\frac{1}{k}, \frac{1}{k}I_\varphi(kx)\right)\right\} \le \max\left(\frac{1}{k_0}, \frac{1}{k_0}I_\Phi(k_0x)\right) \le \frac{1}{k_0}.$$

Since $1/k_0$ can be arbitrarily close to $\|x\|_\varphi$, we get

$$\inf_{k>0}\left\{\max\left(\frac{1}{k}, \frac{1}{k}I_\varphi(kx)\right)\right\} \le \|x\|_\varphi,$$

and the proof is complete.

Second proof. Denote $\|x\| = \inf_{k>0}\{\max(\frac{1}{k}, \frac{1}{k}I_\varphi(kx))\}$.
It is enough to show that $\|x\|_\varphi = 1 \Rightarrow \|x\| = 1$.
Assume that $\|x\|_\varphi = 1$. Then $I_\varphi(x) \le 1$. We can consider two cases.

I. $I_\varphi(x) = 1$. Then, by the convexity of φ, $I_\varphi(kx)/k \le 1$ for all $k \in (0, 1]$ and, hence,

$$\inf_{0<k\le 1} \max\left(\frac{1}{k}, \frac{1}{k}I_\varphi(kx)\right) = \inf_{0<k\le 1}\frac{1}{k} = 1.$$

If $k \ge 1$, then $I_\varphi(kx)/k \ge k \cdot I_\varphi(x)/k = I_\varphi(x) = 1$, so that

$$\inf_{k\ge 1}\left\{\max\left(\frac{1}{k}, \frac{1}{k}I_\varphi(kx)\right)\right\} = \inf_{k\ge 1}\frac{1}{k}I_\varphi(kx) = I_\varphi(x) = 1,$$

because the function $f(k) = I_\varphi(kx)/k$ is nondecreasing on R_+. Hence

$$\|x\| = \min\left\{\inf_{0<k\le 1} \max\left(\frac{1}{k}, \frac{1}{k}I_\varphi(kx)\right), \inf_{k\le 1}\left\{\max\left(\frac{1}{k}, \frac{1}{k}I_\varphi(kx)\right)\right\}\right\} = 1.$$

II. $I_\varphi(x) < 1$. Then $I_\varphi(kx) = \infty$ for any $k > 1$. Otherwise, $I_\varphi(kx) < \infty$ for some $k > 1$ and, by the continuity of the function $g(\lambda) = I_\varphi(\lambda x)$ on $[0, k]$, there is $k_0 \in (1, k)$ such that $I_\varphi(k_0x) = 1$, so that $\|x\|_\varphi = 1/k_0 < 1$, a contradiction. Thus

$$\|x\| = \inf_{0<k\le 1} \max\left(\frac{1}{k}, \frac{1}{k}I_\varphi(kx)\right) = 1,$$

and the proof is complete.

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